NORTH-HOLLAND

On Meet Matrices on Posets

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ABSTRACT

We consider meet matrices on posets as an abstract generalization of greatest common divisor (GCD) matrices. Some of the most important properties of GCD matrices are presented in terms of meet matrices. © *Elsevier Science Inc., 1996*

1. INTRODUCTION

Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. Then the $n \times n$ matrix $[f(x_i, x_j)]$ whose i, j entry is f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j is called the GCD matrix on S associated with f (see e.g. [5, 6]). In 1876, H. J. S. Smith [16] showed that if S is factor-closed, then

$$\det\big[f(x_i, x_j)\big] = (f * \mu)(x_1)(f * \mu)(x_2) \cdots (f * \mu)(x_n),$$

where $*$ is the Dirichlet convolution and μ is the number-theoretic Möbius function. The evaluation $\det[(i, j)] = \phi(1)\phi(2) \cdots \phi(n)$, where ϕ is Euler's totient function, is a famous special case. Since Smith's paper GCD matrices have been studied extensively. For a list of papers, see [9].

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In this paper we consider an abstract generalization of GCD matrices, namely meet matrices on posets. Previously, results in this direction were obtained in [8, 9, 11, 14, 18]. The purpose of this paper is to express some of the most important properties of GCD matrices in the language of meet matrices. To be more precise, we give a structure theorem for meet matrices, then derive explicit expressions and bounds for the determinant of meet matrices, and finally consider the inverse of meet matrices.

2. DEFINITION OF MEET MATRICES

Let $(P, \leqslant)$ be a finite poset. We call P a meet semilattice [17, p. 103] if for any $x, y \in P$ there exists a unique $z \in P$ such that

(1) $z \leqslant x$ and $z \leqslant y$, and
(2) if $w \leqslant x$ and $w \leqslant y$ for some $w \in P$, then $w \leqslant z$.

In such a case z is called the meet of x and y and is denoted by $x \wedge y$.

Let S be a subset of P. We call S lower-closed if for every $x, y \in P$ with $x \in S$ and $y \leqslant x$, we have $y \in S$. We call S meet-closed if for every $x, y \in S$, we have $x \wedge y \in S$. In this case S itself is a meet semilattice.

It is clear that a lower-closed subset of a meet semilattice is always meet-closed, but not conversely. The concepts of "lower-closed" and "meet-closed" are generalizations of "factor-closed" and "gcd-closed" [1, 2], respectively.

In what follows, let P always denote a finite meet semilattice, S a poset that can be embedded in a meet-semilattice, and $\bar{S}$ the unique (up to isomorphism) minimal meet semilattice containing S.

DEFINITION 1. Let $S = \{x_1, x_2, \dots, x_n\}$ be a subset of P, and let f be a function on P with complex values. Then the $n \times n$ matrix $(S)_f = (s_{ij})$, where

$$s_{ij} = f(x_i \wedge x_j),$$

is called the meet matrix on S with respect to f.

3. GENERALIZED TOTIENT FUNCTION

Euler's totient function and its generalizations play an important role in the theory of GCD matrices. We here need the generalization due to Rajarama Bhat [14].

DEFINITION 2. Let $S = \{x_1, x_2, \dots, x_n\}$ be a subset of P, and let f be a function on P with complex values. Then the function $\Psi_{S,f}$ on S is defined inductively by

$$\Psi_{S,f}(x_j) = f(x_j) - \sum_{x_i < x_j} \Psi_{S,f}(x_i), \tag{3.1}$$

where $x_i < x_j$ means that $x_i \leqslant x_j$, $x_i \neq x_j$, or

$$f(x_j) = \sum_{x_i \leqslant x_j} \Psi_{S,f}(x_i). \tag{3.2}$$

REMARK. If S is a factor-closed set of positive integers ordered by divisibility and $f(x) = x$ for all x, then $\Psi_{S,f} = \phi$, Euler's totient function. Thus $\Psi_{S,f}$ in Definition 2 is a generalization of Euler's totient function.

We next derive explicit expressions for $\Psi_{S,f}$ in terms of f.

EXAMPLE 1. Let $S = \{x_1, x_2, \dots, x_n\}$ be meet-closed. Without loss of generality we may assume that $i < j$ whenever $x_i < x_j$. We prove that

$$\Psi_{S,f}(x_j) = \sum_{\substack{z \leqslant x_j \\ x \nleqslant x_t \\ t < j}} \sum_{w \leqslant z} f(w)\mu(w, z), \tag{3.3}$$

where μ is the Möbius function of P.

Proof of (3.3). We have to prove that $\Psi_{S,f}$ given in (3.3) satisfies (3.2), that is,

$$f(x_j) = \sum_{x_i \leqslant x_j} \sum_{\substack{z \leqslant x_i \\ z \nleqslant x_t \\ t < i}} \sum_{w \leqslant z} f(w)\mu(w, z).$$

We write $f(x) = \Sigma_{z \leqslant x} g(z)$ or $g(x) = \Sigma_{z \leqslant x} f(z)\mu(z, x)$ for all $x \in P$. We now have to prove that

$$\sum_{z \leqslant x_j} g(z) = \sum_{x_i \leqslant x_j} \sum_{\substack{z \leqslant x_i \\ z \nleqslant x_t \\ t < i}} g(z). \tag{3.4}$$

It is easy to see that the sums in (3.4) are nonrepetitive, that is, each z is counted only once. Now, consider the sum on the right side of (3.4). Let $x_i \leqslant x_j$ and $z \leqslant x_i$. Then $z \leqslant x_j$. Thus every z occurring on the right side of (3.4) occurs on the left side of (3.4). Conversely, consider the sum on the left side of (3.4). Suppose that $z \leqslant x_j$. Let i be the least number such that $z \leqslant x_i$. Then $z \nleqslant x_t$ for $t < i$. Since S is meet-closed, $x_i \wedge x_j = x_r$ for some $r \leqslant i$. Since $z \leqslant x_i$ and $z \leqslant x_j$, we have $z \leqslant x_r$. By minimality of i, we have $r = i$ or $x_r = x_i$. Therefore $x_r \leqslant x_j$ means that $x_i \leqslant x_j$. Thus every z occurring on the left side of (3.4) occurs on the right side of (3.4). This completes the proof of (3.4), that is, the proof of (3.3). ■

REMARK. Example 1 is a poset-theoretic generalization of Proposition 1 by Beslin and Ligh [2].

EXAMPLE 2. Let $S = \{x_1, x_2, \dots, x_n\}$ be lower-closed. Then application of Möbius inversion (see e.g. [17, p. 116]) yields

$$\Psi_{S,f}(x_j) = \sum_{x_i \leqslant x_j} f(x_i)\mu(x_i, x_j). \tag{3.5}$$

Note that (3.3) reduces to (3.5) when S is lower-closed.

EXAMPLE 3. Let $S = \{x_1, x_2, \dots, x_n\}$ be a chain with $x_1 < x_2 < \cdots < x_n$. Then

$$\Psi_{S,f}(x_1) = f(x_1),$$

$$\Psi_{S,f}(x_j) = f(x_j) - f(x_{j-1}), \qquad j = 2, 3, \dots, n.$$

EXAMPLE 4. Let $S = \{x_1, x_2, \ldots, x_n\}$ be an incomparable set, and let $\bar{S} = \{x_0, x_1, x_2, \ldots, x_n\}$. Then

$$\Psi_{\bar{S},f}(x_0) = f(x_0),$$

$$\Psi_{\bar{S},f}(x_j) = f(x_j) - f(x_0), \qquad j = 1, 2, \ldots, n.$$

Note that we evaluated $\Psi_{\bar{S},f}$ instead of $\Psi_{S,f}$ for the purpose of the corollary to Theorem 3.

4. STRUCTURE THEOREM

For two subsets $S = \{x_1, x_2, \ldots, x_n\}$ and $T = \{y_1, y_2, \ldots, y_m\}$ of P, let $E(S,T) = (e_{ij})$ denote the $n \times m$ incidence matrix defined as

$$e_{ij} = \begin{cases} 1 & \text{if } \; y_j \leqslant x_i, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 1 (Cf. [14, Theorem 12]). *Let $T = \{y_1, y_2, \ldots, y_m\}$ be a meet-closed subset of P containing $S = \{x_1, x_2, \ldots, x_n\}$ $(m \geqslant n)$. Then*

$$(S)_f = E\Lambda E^T = AA^T,$$

where $E = E(S,T)$, $\Lambda = \operatorname{diag}(\Psi_{T,f}(y_1), \ldots, \Psi_{T,f}(y_m))$, and $A = E\Lambda^{1/2}$.

Many such structure theorems have been given in the literature. The structure theorems are very useful in deriving properties for GCD matrices (and more generally for meet matrices). The idea of this factorization is due to Pólya and Szegö [13]. The proof of Theorem 1 is similar in character to the proofs of the previous theorems (see e.g. [1, Theorem 1]). We do not present the details.

5. DETERMINANT OF MEET MATRICES

We recall from the introduction that H. J. S. Smith [16] showed in 1876 that if $S = \{x_1, x_2, \ldots, x_n\}$ is a factor-closed set of positive integers, then

$$\det[f(x_i, x_j)] = (f * \mu)(x_1)(f * \mu)(x_2) \cdots (f * \mu)(x_n).$$

There are a large number of generalizations and analogous of Smith's determinant evaluation in the literature (for a general account, see [9]). Most evaluations assume some restrictions on S and are based on factorizations of the type of Theorem 1. Namely, if S is gcd-closed (or more generally meet-closed) and $T = S$, then the factors in Theorem 1 are square matrices, whose determinants are easy to evaluate. We here apply Theorem 1 to evaluate the determinant of meet matrices on meet-closed sets: see Theorem 2.

There is also another way to apply Theorem 1 to obtain determinant evaluations, namely the use of the Cauchy-Binet formula [7]. We may then assume S to be an arbitrary subset of P. This idea was first applied for GCD matrices by Li [10, Theorem 2]. We here apply this idea for meet matrices: see Theorem 3.

THEOREM 2 [14, Theorem 13]. *If S is meet-closed, then*

$$\det(S)_f = \prod_{i=1}^{n} \Psi_{S,f}(x_i).$$

Proof. Take $T = S$ in Theorem 1, and arrange the elements of S so that E is a triangular matrix (whose diagonal elements are equal to 1). ∎

COROLLARY 1. *If S is meet-closed, then*

$$\det(S)_f = \prod_{i=1}^{n} \sum_{\substack{z \leqslant x_i \\ z \nleqslant x_t \\ t<i}} \sum_{w \leqslant z} f(w)\mu(w,z).$$

COROLLARY 2. *If S is lower-closed, then*

$$\det(S)_f = \prod_{i=1}^{n} \sum_{x_k \leqslant x_i} f(x_k)\mu(x_k, x_i).$$

COROLLARY 3. *If $S = \{x_1, x_2, \ldots, x_n\}$ is a chain with $x_1 < x_2 < \cdots < x_n$, then*

$$\det(S)_f = f(x_1)\prod_{i=2}^{n}[f(x_i) - f(x_{i-1})].$$

Corollaries 1–3 follow from Theorem 2 and Examples 1–3 (Section 3).

For any $n \times m$ matrix M with $m \geqslant n$, we denote by $M(k_1, k_2, \dots, k_n)$ the $n \times n$ submatrix of M which contains the columns $1 \leqslant k_1 < k_2 < \cdots < k_n \leqslant m$.

THEOREM 3. *Let $T = \{y_1, y_2, \dots, y_m\}$ be a meet-closed subset of P containing $S = \{x_1, x_2, \dots, x_n\}$. Then*

$$\det(S)_f = \sum_{1 \leqslant k_1 < k_2 < \cdots < k_n \leqslant m} \det[E(k_1, k_2, \dots, k_n)]^2$$

$$\times \Psi_{T,f}(y_{k_1}) \Psi_{T,f}(y_{k_2}) \cdots \Psi_{T,f}(y_{k_n}),$$

where $E = E(S, T)$.

Proof. Theorem 3 is a direct consequence of Theorem 1 and the Cauchy-Binet formula [7]. ■

COROLLARY. *If S is incomparable, then*

$$\det(S)_f = f(x_0)[f(x_1) - f(x_0)] \cdots [f(x_n) - f(x_0)]$$

$$\times \left(\frac{1}{f(x_0)} + \frac{1}{f(x_1) - f(x_0)} + \cdots + \frac{1}{f(x_n) - f(x_0)} \right)$$

(provided that the appropriate denominators are nonzero).

Proof. This is a consequence of Theorem 3 and Example 4 (Section 3).

REMARK. The above corollary is a poset-theoretic generalization of a formula by Beslin and Ligh [3, § 4, Case 2].

6. LOWER BOUNDS FOR $\det(S)_f$

Throughout the rest of the paper, let f be a function on P with real values.

Bourque and Ligh [6] proved that if $S = \{x_1, x_2, \ldots, x_n\}$ is a set of distinct positive integers and if $(f * \mu)(d) > 0$ for all $d \in \{d : d \mid x,\ x \in S\}$, then

$$\det[f(x_i, x_j)] \geqslant \prod_{k=1}^{n} (f * \mu)(x_k) \tag{6.1}$$

and the equality holds if and only if S is factor-closed. This result was first proved in the case $f(x) = x$ for all x by Li [10, Theorem 3]. In this section we express this result in a more general setting and we also improve the lower bound for $\det[f(x_i, x_j)]$ given in (6.1): see (6.3).

THEOREM 4. *Let $T = \{y_1, y_2, \ldots, y_m\}$ be a meet-closed set containing $S = \{x_1, x_2, \ldots, x_n\}$, and let $\sum_{w \leqslant z} f(w)\mu(w, z) > 0$ for all $z \in \{z : z \leqslant y,\ y \in T\}$. Then*

$$\det(S)_f \geqslant \prod_{k=1}^{n} \Psi_{T,f}(x_k), \tag{6.2}$$

and the equality holds if and only if S is meet-closed and $\forall x \in S : \forall z \leqslant x : z \notin T \setminus S$.

We adapt the proof of [14, Theorem 10] for GCD matrices. This proof applies a result due to Minkowski (see [12]). Theorem 4 could also be proved without Minkowski's result using some elementary manipulations. We do not present these details here.

REMARK (Minkowski). Let C and D be $n \times n$ real symmetric matrices. If C is positive definite and D is positive semidefinite, then

$$\det(C + D) \geqslant \det C + \det D,$$

and the equality holds if and only if $D = 0$.

Proof of Theorem 4. Without loss of generality, we may assume that $x_i < x_j$ implies $i < j$ and that $y_i = x_i$ for $1 \leqslant i \leqslant n$. Partition $E = E(S, T)$ as $E = [E_1 \mid E_2]$, where E_1 is an $n \times n$ matrix and E_2 is an $n \times (m - n)$ matrix. Let Λ_1 be the $n \times n$ diagonal matrix whose rth diagonal entry is $\Psi_{T,f}(y_r)$, and let Λ_2 be the $(m - n) \times (m - n)$ diagonal matrix whose rth diagonal entry is $\Psi_{T,f}(y_{n+r})$. Let $C = E_1 \Lambda_1 E_1^T$ and $D = E_2 \Lambda_2 E_2^T$. Then, by

Theorem 1, $(S)_f = C + D$. Now we show that C is positive definite and D is positive semidefinite. Under the assumptions of Theorem 4, $\Psi_{T,f}(y) > 0$ for all $y \in T$ (see Example 1, Section 3). We thus have the decompositions

$$C = (E_1\Lambda_1^{1/2})(E_1\Lambda_1^{1/2})^T$$

and

$$D = (E_2\Lambda_2^{1/2})(E_2\Lambda_2^{1/2})^T,$$

which show the C and D are positive semidefinite. Further,

$$\det C = \prod_{k=1}^{n} \Psi_{T,f}(x_k) > 0;$$

hence C is positive definite. Now, applying Minkowski's result, we obtain

$$\det(S)_f = \det(C + D) \geqslant \det C + \det D \geqslant \det C = \prod_{k=1}^{n} \Psi_{T,f}(x_k),$$

which shows (6.2). The equality holds if and only if $D = 0$. As $\Lambda_2 > 0$, $D = 0$ if and only if $E_2 = 0$, that is, if and only if $\forall x \in S : \forall z \leqslant x : z \notin T \setminus S$. But then the condition that T is meet-closed forces S to be meet-closed. This completes the proof of Theorem 4. ∎

COROLLARY 1. *Let $T = \{y_1, y_2, \dots, y_m\}$ be a lower-closed set containing $S = \{x_1, x_2, \dots, x_n\}$, and let $\sum_{w \leqslant z} f(w)\mu(w, z) > 0$ for all $z \in \{z : z \leqslant y, y \in T\}$. Then*

$$\det(S)_f \geqslant \prod_{k=1}^{n} \Psi_{T,f}(x_k),$$

and the equality holds if and only if S is lower-closed.

Proof. Corollary 1 is a direct consequence of Theorem 4. ∎

REMARK. If S is a set of positive integers and T is the minimal factor-closed set containing S, then Corollary 1 reduces to the result of Bourque and Ligh [6] mentioned in the beginning of this section.

Corollary 2. *Let $S = \{x_1, x_2, \ldots, x_n\}$, and let $\sum_{w \leqslant z} f(w)\mu(w, z) > 0$ for all $z \in \{z : z \leqslant y,\ y \in S\}$. Then*

$$\det(S)_f \geqslant \prod_{k=1}^{n} \Psi_{\bar{S}, f}(x_k),$$

and the equality holds if and only if S is meet-closed.

Proof. It is easy to see that $\{z : z \leqslant x,\ x \in S\} = \{z : z \leqslant x,\ x \in \bar{S}\}$. Thus, taking $T = \bar{S}$ in Theorem 4, we obtain Corollary 2. ■

Corollary 3. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let $(f * \mu)(d) > 0$ for all $d \in \{d : d \mid x,\ x \in S\}$. Then*

$$\det[f(x_i, x_j)] \geqslant \prod_{k=1}^{n} \sum_{\substack{d \mid x_k \\ d \nmid y_t \\ y_t < x_k}} (f * \mu)(d), \tag{6.3}$$

where $\bar{S} = \{y_1, y_2, \ldots, y_m\}$, and the equality holds if and only if S is gcd-closed.

Proof. Corollary 3 is a consequence of Corollary 2 and Example 1 (Section 3). ■

Remark. Note that (6.3) improves the lower bound (6.1).

7. UPPER BOUNDS FOR $\det(S)_f$

Li [10, Theorem 1] proved for the basic GCD matrix the inequality

$$\det[(x_i, x_j)] \leqslant x_1 x_2 \cdots x_n - \frac{n!}{2}.$$

Bourque and Ligh [6] proved for GCD matrices associated with arithmetical functions a weaker inequality. In fact, they proved that if $(f * \mu)(d) > 0$ for all $d \in \{d : d \mid x,\ x \in S\}$, then $(S)_f$ is positive definite and thus

$$\det[f(x_i, x_j)] \leqslant f(x_1) f(x_2) \cdots f(x_n).$$

We here show that this result of Bourque and Ligh also holds for meet matrices. (Note that the proof of Bourque and Ligh contains some errors.)

THEOREM 5. *Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct elements of P, and let $\sum_{w \leqslant z} f(w)\mu(w, z) > 0$ for all $z \in \{z : z \leqslant x,\ x \in S\}$. Then $(S)_f$ is positive definite.*

Proof. Application of Corollary 2 of Theorem 4 and Example 1 (Section 3) shows that $\det(S)_f > 0$. The matrix $(S)_f$ is thus positive definite. ∎

COROLLARY. *Under the assumptions of Theorem 5,*

$$\det(S)_f \leqslant f(x_1)f(x_2)\cdots f(x_n).$$

8. INVERSE OF $(S)_f$

It follows from Theorem 5 that $\det(S)_f > 0$ for any subset S of P and any real-valued function f of P satisfying $\sum_{w \leqslant z} f(w)\mu(w, z) > 0$ for all $z \in \{z : z \leqslant x,\ x \in S\}$. Thus any meet matrix $(S)_f$ under the above condition on f is invertible. In this section we calculate the inverse of the meet matrix $(S)_f$ on lower-closed sets. This is a generalization of the corresponding result for GCD matrices on factor-closed sets given by Bourque and Ligh [4, Theorem 1; 6, Corollary 1].

We also point out a difficulty that arises in calculating the inverse of meet matrices on sets which are not lower-closed. Overcoming this difficulty is left as an open problem. Note that Bourque and Ligh [4, 6] calculated the inverse of GCD matrices on gcd-closed sets.

THEOREM 6. *Let $S = \{x_1, x_2, \dots, x_n\}$ be a lower-closed subset of P, and let $\Psi_{S,f}(x_j) \equiv \sum_{x_i \leqslant x_j} f(x_i)\mu(x_i, x_j) \neq 0$ for all $x_j \in S$. Then $(S)_f$ is invertible and $(S)_f^{-1} = (b_{ij})$, where*

$$b_{ij} = \sum_{\substack{x_i \leqslant x_k \\ x_j \leqslant x_k}} \frac{1}{\Psi_{S,f}(x_k)} \mu(x_i, x_k)\mu(x_j, x_k),$$

μ being the Möbius function of P.

Proof. The matrix $(S)_f$ is invertible by Corollary 2 of Theorem 2. It is clear that $E \equiv E(S, S) = [\zeta(x_i, x_j)]^T$, where ζ is the zeta function of P (see [17, p. 114]). Thus $E^{-1} = [\mu(x_i, x_j)]^T \equiv U^T$. By Theorem 1, $(S)_f = E\Lambda E^T$, where $\Lambda = \mathrm{diag}(\Psi_{S,f}(x_1), \ldots, \Psi_{S,f}(x_n))$. Thus $(S)_f^{-1} = U\Lambda^{-1}U^T$. We thus arrive at Theorem 6. ■

REMARK. Note that if S is not lower-closed, then $E^{-1} \neq U^T$. For example, if $S = \{1, 4\}$ is a set of positive integers, then

$$E = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$U = \begin{pmatrix} \mu(1) & 0 \\ \mu(4) & \mu(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where μ is the number-theoretic Möbius function. Note that if S is meet-closed, then we could consider the zeta function ζ_S and the Möbius function μ_S of S and apply Theorem 1 as in Theorem 6.

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